N6 • Developing proportional reasoning

Mathematical goals
To help learners to:
- reflect on the reasoning they currently use when solving proportion problems;
- examine proportion problems and appreciate their multiplicative structure;
- create their own variants of proportion problems.

Starting points
Proportional reasoning is notoriously difficult for many learners. Many have difficulty in recognising the multiplicative structures that underlie proportion problems. Instead, they use addition methods, or informal methods using doubling, halving and adding. This session aims to expose and build on this prior learning.

Learners are given four direct proportion problems to solve, taken from different areas of the mathematics curriculum. They then compare their methods for solving these with methods produced by other learners. This leads to a discussion that compares the use of more primitive informal methods that use adding, doubling and halving with the use of more sophisticated methods that use multiplication.

Materials required
- OHT 1 – Solving proportion problems in one step;
- OHT 2 – Solving proportion problems in two steps;
- an OHT of ‘Paint prices’ from Sheet 1 – Problems (page 1).

For each learner you will need:
- Sheet 1 – Problems (pages 1 and 2);
- Sheet 2 – Sample work;
- Sheet 3 – Making up your own questions;
- a calculator.

Time needed
At least 1 hour. If time is short, fewer problems may be tackled.
**Suggested approach**  

**Beginning the session**

Give out copies of Sheet 1 – *Problems* (pages 1 and 2). Allow learners time to tackle the problems individually or in pairs. They may use calculators if they wish. Do not comment on learners’ solution strategies at this stage as the purpose of the session is to compare different strategies.

**Working in groups**

Hand out copies of Sheet 2 – *Sample work*. Invite learners to assess these pieces of work.

They should attempt to:

- correct the work;
- write advice to the learner, identifying and explaining the errors that have been made and how the solution strategies can be improved.

The pieces of work illustrate the following issues:

**Recipe**

The learner has answered part 1 correctly, but part 2 incorrectly. For both parts, the learner has adopted an addition strategy. In part 1, she has reasoned that, because $10 = 4 + 4 + \text{half of } 4$, then the quantities in the recipe may be deduced by doubling, then adding one half as much again. This is a correct strategy and is one that is helpful when working out calculations mentally. When a calculator is available, however, it is not the most efficient method.

In the second problem, the learner has followed the same strategy but has then added ‘tops and bottoms’.

**Paint prices**

In part 1 the learner has chosen, wrongly, to divide rather than to multiply. This may be because the learner believes that ‘division will make numbers smaller’ and so $15 \div 0.6$ will be less than 15. (Multiplication has been rejected on the grounds that it will result in a number greater than 15.)

In parts 2 and 3, a ‘halving and adding’ strategy has been correctly used.

Part 4 is correct; a multiplying strategy has been correctly used.
Enlarging a photograph to make a poster

In these two commonly-found answers, an additive strategy has been incorrectly used.

Advertising

The percentages have been correctly calculated using an addition strategy. With a calculator, a more efficient strategy would have been to simply calculate the missing entries by multiplying or dividing by 3.6.

Whole group discussion

Place an OHT of the ‘Paint prices’ problem from Sheet 1 – Problems onto the overhead projector and ask learners to share their comments about the sample work. Invite learners to suggest better ways of doing the problems, assuming that a calculator is available. In particular, challenge them to find a simple and efficient way of obtaining each answer using a calculator. For example, the missing paint prices can be found quite easily by multiplying every capacity by 15.

Ask learners if they noticed that the problems all have something in common.

Explain the following ideas, asking learners to contribute to each step. Write their responses on the board.

All the problems we have looked at in this session have involved two quantities. Can you help me list them?

- Number of pancakes
- Quantity of paint
- Height of poster
- Percentage of money spent

- Amount of ingredients
- Cost of paint
- Width of poster
- Angle in the pie chart

These are proportional situations. If we double the first quantity, we double the second. If we plotted a graph of the two quantities, we would get a straight line through the origin.

Can you suggest some more pairs of quantities that are proportional?

Can you suggest some that are not?

Learners may suggest other proportional pairs such as distance travelled and time taken (assuming constant speed), and others that are not, such as speed of travel and time taken (this is inversely proportional).
Show OHT 1 – *Solving proportion problems in one step* (or draw the diagram on the board). Explain that you will show a method for working out proportional problems using this diagram.

Look at the paint problem. The two quantities are amount bought (litres) and cost (£). 1 litre costs £15. (Fill in the top two boxes in the diagram). We need to know how much 0.6 litres costs. (Fill in the remaining quantities in the diagram, explaining the reasoning.)

Explain how proportion problems are really just multiplication problems and show how the multipliers may be found and used. The vertical multipliers are dimensionless. The horizontal ones are rates (here £ per litre).

Create a new problem by changing the numbers, for example as shown here. Ask learners to state the new problem in their own words and to suggest the multipliers which, this time, are not so obvious. Some may see that the rows involve multiplying by $\frac{10}{4}$ or 2.5, and the columns by $\frac{7}{4}$ or 1.75.

In the same way, create additional problems using different sets of numbers, some of which use multipliers that are less than 1.

Learners who struggle may like to use two steps (using OHT 2 – *Solving proportion problems in two steps*), first dividing by 4 to find the price of 1 litre, then multiplying by 7 to find the price of 7 litres, as shown here.

Ask learners to return to the original problems and, using this diagram method, to solve any they could not do earlier.
Reviewing and extending learning

Ask learners to tell you how they can recognise when a situation is a direct proportion and when it is not.

- When one value is zero, so is the other.
- When one variable doubles, so does the other.
- The graph of one variable against the other is a straight line through the origin.
- The formula is \( y = ax \).

What learners might do next

Issue learners with Sheet 3 – Making up your own questions. This sheet contains a selection of problems with numbers missing. Learners need to:

- decide which situations are direct proportions and which are not;
- write their own numbers in the spaces;
- solve the problems that are created, using both their own informal methods and the diagram method.

Encourage learners to use at least two sets of numbers for each problem. One set should make the problem quite easy (but not trivial), and one set should make it quite difficult.

If some learners are more able, ask them to write variable names for each space (e.g. \( x \), \( y \) and \( z \)) and to write solutions for each situation using these letters.

Further ideas

This session involves analysing and comparing different methods for solving problems. This is a powerful idea that may be used in any topic. Why not give your learners a completed examination paper to mark and comment on? This is very useful if the answers that you provide reveal common misconceptions.
RECIPE

Pancake mix
For 4 pancakes you will need:
6 dessertspoons of flour
$\frac{1}{4}$ litre milk
1 pinch salt
1 egg

You want to make 10 pancakes.
1. How much flour do you need?
2. How much milk do you need?

PAINT PRICES

Calculate the missing prices of the paint cans below.
The prices are proportional to the amount of paint in the can.

<table>
<thead>
<tr>
<th>Price</th>
<th>0.6 litres</th>
<th>0.75 litres</th>
<th>1 litre</th>
<th>2.5 litres</th>
<th>4.54 litres (1 gallon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>£ ...</td>
<td>£ ...</td>
<td>£15</td>
<td>£ ...</td>
<td>£ ...</td>
<td>£ ...</td>
</tr>
</tbody>
</table>
ENLARGING A PHOTOGRAPH TO MAKE A POSTER

1. The photograph is enlarged to make a poster. The photograph is 10 cm wide and 16 cm high. The poster is 25 cm wide. How high is the poster?

2. In the poster, the building is 30 cm tall. How tall is it in the photograph?

ADVERTISING

<table>
<thead>
<tr>
<th>% spent</th>
<th>Angle in pie chart</th>
</tr>
</thead>
<tbody>
<tr>
<td>Press</td>
<td>55</td>
</tr>
<tr>
<td>Television</td>
<td>20  72°</td>
</tr>
<tr>
<td>Posters</td>
<td>9°</td>
</tr>
<tr>
<td>Other</td>
<td></td>
</tr>
</tbody>
</table>

The pie chart shows the proportion spent on advertising in different media in one year. Calculate the missing entries in the table.
N6 Sheet 2 – Sample work

(i) Mark the answers right or wrong.
(ii) Find the causes of the mistakes.
(iii) Write down your advice to the learner, explaining how the work should be improved, even when the answer is right.

Recipe

1. 10 pancakes = 6 + 6 + 3 = 15 spoons of flour
2. = ¼ + ¼ + ½ = 3/16 pints of milk.

Paint prices
1. 0.6 litres = 15 ÷ 0.6 = 25p
2. 0.75 litres = ½ + ¼
   = 3/4 = 7.50 + 3.75 = £11.25
3. 2.5 litres = 2 + ½
   = £30 + £7.50 = £37.50
4. 4.54 litres = 15 × 4.54
   = 68.1

Enlarging a photograph to make a poster

The poster is 15 cm bigger

1. 16 + 15 = 31 cm high.
2. 30 - 15 = 15 cm high.

Advertising

Press = 72° + 72° + 36° + 18° = 198°
Posters = 9° + 2.5%
Other = 100 - 77.5 = 22.5%
   = 20% + 2.5%
   ⇒ 72° + 9° = 81°
N6 OHT 1 – *Solving proportion problems in one step*
N6 OHT 2 – Solving proportion problems in two steps
## N6 Sheet 3 – *Making up your own questions*

<table>
<thead>
<tr>
<th>CYCLE</th>
<th>PETROL</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Bicycle" /></td>
<td><img src="image" alt="Petrol can" /></td>
</tr>
<tr>
<td>It takes ....... minutes to cycle ...... miles.</td>
<td>..... litres cost £ .....</td>
</tr>
<tr>
<td>At the same speed, how long does it take to cycle ...... miles?</td>
<td>How much will ...... litres cost?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TRIANGLES</th>
<th>DRIVING</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Triangles" /></td>
<td><img src="image" alt="Car" /></td>
</tr>
<tr>
<td>These two triangles are similar.</td>
<td>If I drive at ...... miles per hour, the journey will take ...... hours.</td>
</tr>
<tr>
<td>Calculate the length marked x.</td>
<td>How long will it take if I drive at ...... miles per hour?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>LINE</th>
<th>FIRE</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Line" /></td>
<td><img src="image" alt="Fire escapes" /></td>
</tr>
<tr>
<td>A straight line passes through the points (0,0) and ( ......, ......).</td>
<td>It would take ..... minutes to vacate a building if we put in ..... fire escapes.</td>
</tr>
<tr>
<td>It also passes through the point ( ......, y').</td>
<td>How long would it take with ..... fire escapes?</td>
</tr>
<tr>
<td>Calculate the value of y.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MONEY</th>
<th>MAP</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Money" /></td>
<td><img src="image" alt="Map" /></td>
</tr>
<tr>
<td>£ ..... is worth the same as ...... dollars.</td>
<td>A road ...... cm long on a map is ...... km long in real life.</td>
</tr>
<tr>
<td>If I change £ ......, how many dollars will I get?</td>
<td>A river is ...... cm long on the map. How long is the real river?</td>
</tr>
</tbody>
</table>