Improving learning in mathematics: challenges and strategies

Swan-ImprovingLearningInMaths.pdf

Some underlying principles

(i) Build on the knowledge learners bring to sessions
(ii) Expose and discuss common misconceptions
(iii) Develop effective questioning
(iv) Use cooperative small group work
(v) Emphasise methods rather than answers
(vi) Use rich collaborative tasks
(vii) Create connections between mathematical topics
(viii) Use technology in appropriate ways

A 'Transmission' view		'Connected', 'challenging' view
A given body of knowledge and standard procedures that has to be 'covered'.	**Mathematics is ...** ⟷	An interconnected body of ideas and reasoning processes.
An individual activity based on watching, listening and imitating until fluency is attained.	**Learning is ...** ⟷	A collaborative activity in which learners are challenged and arrive at understanding through discussion.
Structuring a linear curriculum for learners. Giving explanations before problems. Checking that these have been understood through practice exercises. Correcting misunderstandings.	**Teaching is ...** ⟷	Exploring meaning and connections through non-linear dialogue between teacher and learners. Presenting problems before offering explanations. Making misunderstandings explicit and learning from them.

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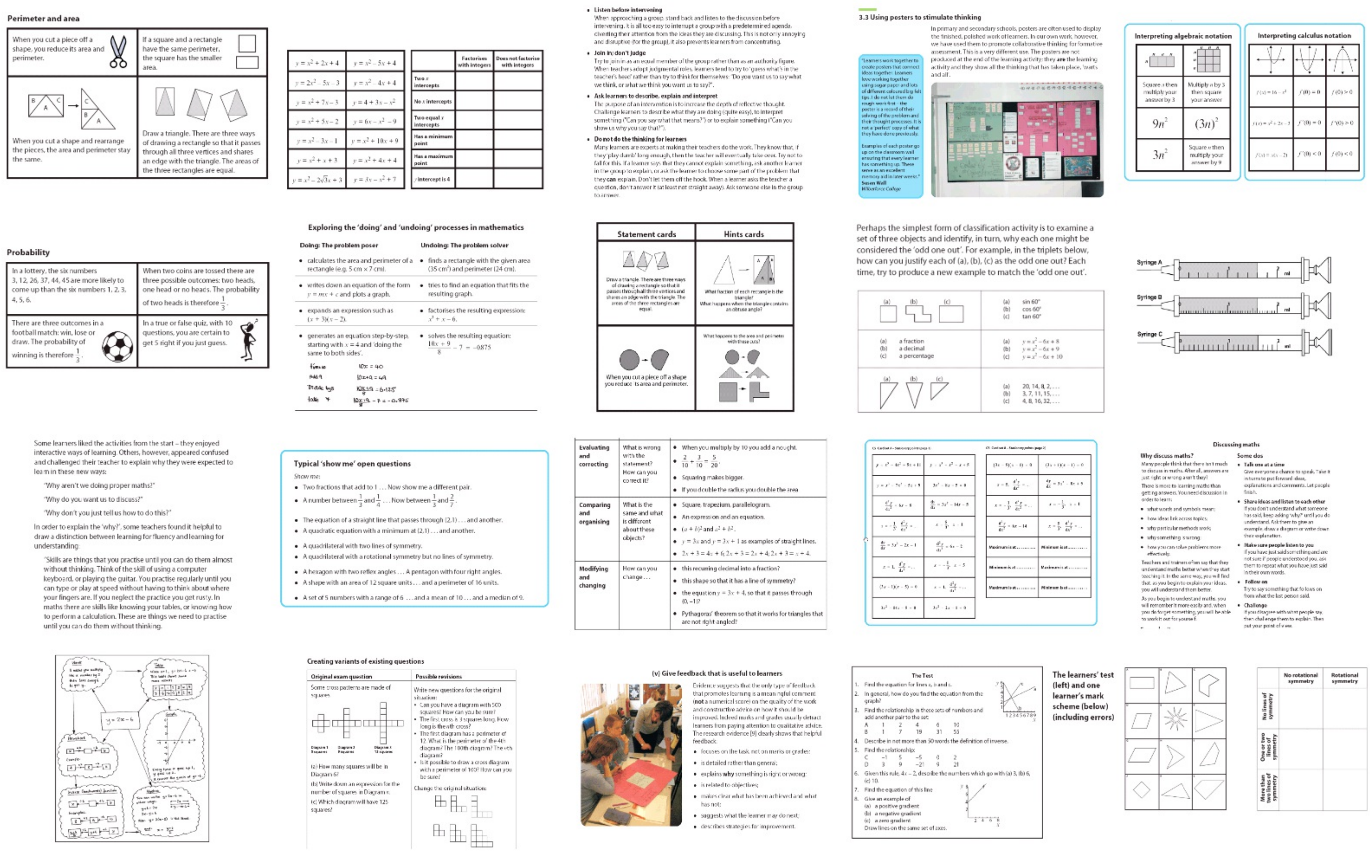

Typical 'show me' open questions

Show me:

- Two fractions that add to 1 . . . Now show me a different pair.
- A number between $\frac{1}{3}$ and $\frac{1}{4}$ . . . Now between $\frac{1}{3}$ and $\frac{2}{7}$.
- The equation of a straight line that passes through (2,1) . . . and another.
- A quadratic equation with a minimum at (2,1) . . . and another.
- A quadrilateral with two lines of symmetry.
- A quadrilateral with a rotational symmetry but no lines of symmetry.
- A hexagon with two reflex angles . . . A pentagon with four right angles.
- A shape with an area of 12 square units . . . and a perimeter of 16 units.
- A set of 5 numbers with a range of 6 . . . and a mean of 10 . . . and a median of 9.

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3.3 Using posters to stimulate thinking

In primary and secondary schools, posters are often used to display the finished, polished work of learners. In our own work, however, we have used them to promote collaborative thinking for formative assessment. This is a very different use. The posters are not produced at the end of the learning activity; they **are** the learning activity and they show all the thinking that has taken place, 'warts and all'.

"Learners work together to create posters that connect ideas together. Learners love working together using sugar paper and lots of different coloured big felt tips. I do not let them do rough work first – the poster is a record of their solving of the problem and their thought processes. It is not a 'perfect' copy of what they have done previously.

Examples of each poster go up on the classroom wall ensuring that every learner has something up. These serve as an excellent memory aid in later weeks."
Susan Wall
Wilberforce College

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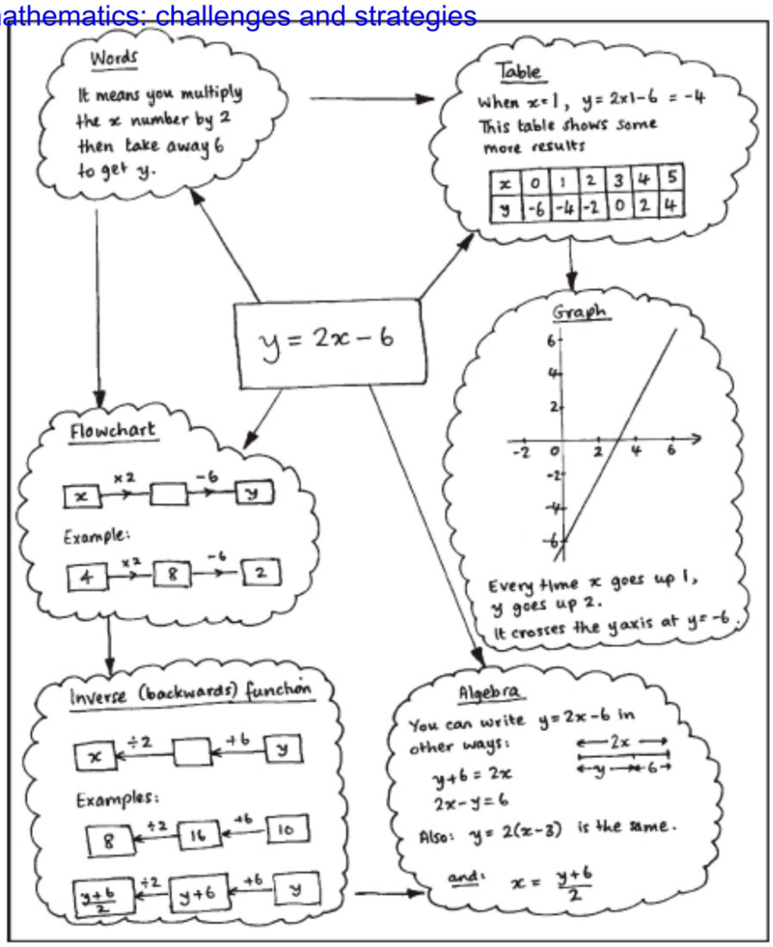
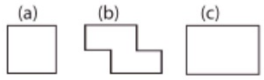
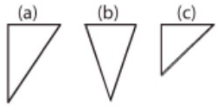


Perhaps the simplest form of classification activity is to examine a set of three objects and identify, in turn, why each one might be considered the ‘odd one out’. For example, in the triplets below, how can you justify each of (a), (b), (c) as the odd one out? Each time, try to produce a new example to match the ‘odd one out’.

(a) (b) (c) 	(a) sin 60° (b) cos 60° (c) tan 60°
(a) a fraction (b) a decimal (c) a percentage	(a) $y = x^2 - 6x + 8$ (b) $y = x^2 - 6x + 9$ (c) $y = x^2 - 6x + 10$
(a) (b) (c) 	(a) 20, 14, 8, 2, . . . (b) 3, 7, 11, 15, . . . (c) 4, 8, 16, 32, . . .

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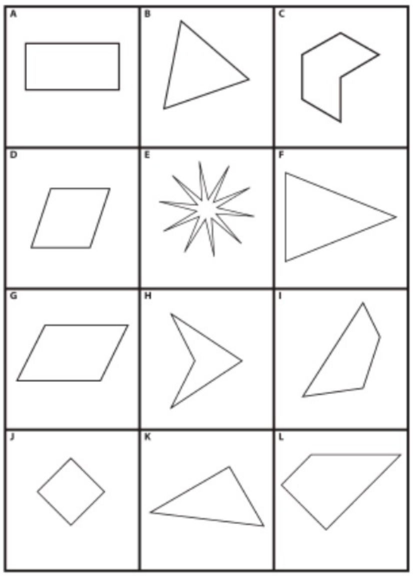

	No rotational symmetry	Rotational symmetry
No lines of symmetry		
One or two lines of symmetry		
More than two lines of symmetry		

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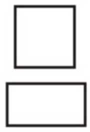
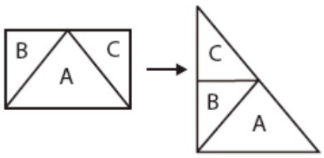
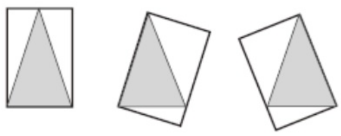
$y = x^2 + 2x + 4$	$y = x^2 - 5x + 4$
$y = 2x^2 - 5x - 3$	$y = x^2 - 4x + 4$
$y = x^2 + 7x - 3$	$y = 4 + 3x - x^2$
$y = x^2 + 5x - 2$	$y = 6x - x^2 - 9$
$y = x^2 - 3x - 1$	$y = x^2 + 10x + 9$
$y = x^2 + x + 3$	$y = x^2 + 4x + 4$
$y = x^2 - 2\sqrt{3}x + 3$	$y = 3x - x^2 + 7$

	Factorises with integers	Does not factorise with integers
Two x intercepts		
No x intercepts		
Two equal x intercepts		
Has a minimum point		
Has a maximum point		
y intercept is 4		

Perimeter and area

When you cut a piece off a shape, you reduce its area and perimeter. 	If a square and a rectangle have the same perimeter, the square has the smaller area. 
 When you cut a shape and rearrange the pieces, the area and perimeter stay the same.	 Draw a triangle. There are three ways of drawing a rectangle so that it passes through all three vertices and shares an edge with the triangle. The areas of the three rectangles are equal.

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Probability

In a lottery, the six numbers 3, 12, 26, 37, 44, 45 are more likely to come up than the six numbers 1, 2, 3, 4, 5, 6.	When two coins are tossed there are three possible outcomes: two heads, one head or no heads. The probability of two heads is therefore $\frac{1}{3}$.
There are three outcomes in a football match: win, lose or draw. The probability of winning is therefore $\frac{1}{3}$. 	In a true or false quiz, with 10 questions, you are certain to get 5 right if you just guess. 

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Exploring the 'doing' and 'undoing' processes in mathematics

Doing: The problem poser	Undoing: The problem solver
• calculates the area and perimeter of a rectangle (e.g. 5 cm × 7 cm).	• finds a rectangle with the given area (35 cm^2) and perimeter (24 cm).
• writes down an equation of the form $y = mx + c$ and plots a graph.	• tries to find an equation that fits the resulting graph.
• expands an expression such as $(x + 3)(x - 2)$.	• factorises the resulting expression: $x^2 + x - 6$.
• generates an equation step-by-step, starting with $x = 4$ and 'doing the same to both sides'. times 10 $\quad 10x = 40$ Add 9 $\quad 10x + 9 = 49$ Divide by 8 $\quad \frac{10x+9}{8} = 6.125$ take 7 $\quad \frac{10x+9}{8} - 7 = -0.875$	• solves the resulting equation: $\frac{10x + 9}{8} - 7 = -0.875$

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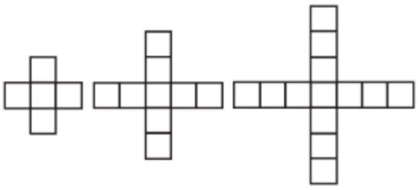
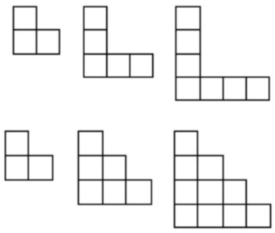
Creating variants of existing questions

Original exam question	Possible revisions
Some cross patterns are made of squares.  **Diagram 1** **5 squares** **Diagram 2** **9 squares** **Diagram 3** **13 squares** (a) How many squares will be in Diagram 6? (b) Write down an expression for the number of squares in Diagram n. (c) Which diagram will have 125 squares?	Write new questions for the original situation: • Can you have a diagram with 500 squares? How can you be sure? • The first cross is 3 squares long. How long is the nth cross? • The first diagram has a perimeter of 12. What is the perimeter of the 4th diagram? The 100th diagram? The nth diagram? • Is it possible to draw a cross diagram with a perimeter of 100? How can you be sure? Change the original situation: 

C5 Card set A – *Stationary points* (page 1)

$y = x^3 - 4x^2 + 5x + 11$	$y = x^3 - x^2 - x + 5$
$y = x^3 - 7x^2 - 5x + 9$	$3x^2 - 8x + 5 = 0$
$\frac{d^2y}{dx^2} = 6x - 8$	$\frac{dy}{dx} = 3x^2 - 14x - 5$
$x = -\frac{1}{3}, \quad \frac{d^2y}{dx^2} = \ldots$	$x = \frac{5}{3}, \quad x = 1$
$\frac{dy}{dx} = 3x^2 - 2x - 1$	$\frac{d^2y}{dx^2} = 6x - 2$
$x = 1, \quad \frac{d^2y}{dx^2} = \ldots$	$x = -\frac{1}{3}, \quad x = 5$
$(3x + 1)(x - 5) = 0$	$x = 1, \quad \frac{d^2y}{dx^2} = \ldots$
$3x^2 - 14x - 5 = 0$	$3x^2 - 2x - 1 = 0$

C5 Card set A – *Stationary points* (page 2)

$(3x - 5)(x - 1) = 0$	$(3x + 1)(x - 1) = 0$
$x = 5, \quad \frac{d^2y}{dx^2} = \ldots$	$\frac{dy}{dx} = 3x^2 - 8x + 5$
$x = -\frac{1}{3}, \quad \frac{d^2y}{dx^2} = \ldots$	$x = -\frac{1}{3}, \quad x = 1$
$\frac{d^2y}{dx^2} = 6x - 14$	$x = \frac{5}{3}, \quad \frac{d^2y}{dx^2} = \ldots$
Maximum is at	**Minimum is at ....**
Minimum is at	**Maximum is at ....**
Maximum is at	**Minimum is at ....**

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Some learners liked the activities from the start – they enjoyed interactive ways of learning. Others, however, appeared confused and challenged their teacher to explain why they were expected to learn in these new ways:

> "Why aren't we doing proper maths?"
>
> "Why do you want us to discuss?"
>
> "Why don't you just tell us how to do this?"

In order to explain the 'why?', some teachers found it helpful to draw a distinction between learning for fluency and learning for understanding:

> "Skills are things that you practise until you can do them almost without thinking. Think of the skill of using a computer keyboard, or playing the guitar. You practise regularly until you can type or play at speed without having to think about where your fingers are. If you neglect the practice you get rusty. In maths there are skills like knowing your tables, or knowing how to perform a calculation. These are things we need to practise until you can do them without thinking.

Discussing maths

Why discuss maths?

Many people think that there isn't much to discuss in maths. After all, answers are just right or wrong aren't they?

There is more to learning maths than getting answers. You need discussion in order to learn:

- what words and symbols mean;
- how ideas link across topics;
- why particular methods work;
- why something is wrong;
- how you can solve problems more effectively.

Teachers and trainers often say that they understand maths better when they start teaching it. In the same way, you will find that, as you begin to explain your ideas, you will understand them better.

As you begin to understand maths, you will remember it more easily and, when you do forget something, you will be able to work it out for yourself.

Some don'ts

Some dos

- **Talk one at a time**
  Give everyone a chance to speak. Take it in turns to put forward ideas, explanations and comments. Let people finish.
- **Share ideas and listen to each other**
  If you don't understand what someone has said, keep asking 'why?' until you do understand. Ask them to give an example, draw a diagram or write down their explanation.
- **Make sure people listen to you**
  If you have just said something and are not sure if people understood you, ask them to repeat what you have just said in their own words.
- **Follow on**
  Try to say something that follows on from what the last person said.
- **Challenge**
  If you disagree with what people say, then challenge them to explain. Then put your point of view.

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Evaluating and correcting	What is wrong with the statement? How can you correct it?	• When you multiply by 10 you add a nought. • $\frac{2}{10} + \frac{3}{10} = \frac{5}{20}$. • Squaring makes bigger. • If you double the radius you double the area.
Comparing and organising	What is the same and what is different about these objects?	• Square, trapezium, parallelogram. • An expression and an equation. • $(a + b)^2$ and $a^2 + b^2$. • $y = 3x$ and $y = 3x + 1$ as examples of straight lines. • $2x + 3 = 4x + 6$; $2x + 3 = 2x + 4$; $2x + 3 = x + 4$.
Modifying and changing	How can you change . . .	• this recurring decimal into a fraction? • this shape so that it has a line of symmetry? • the equation $y = 3x + 4$, so that it passes through $(0, -1)$? • Pythagoras' theorem so that it works for triangles that are not right-angled?

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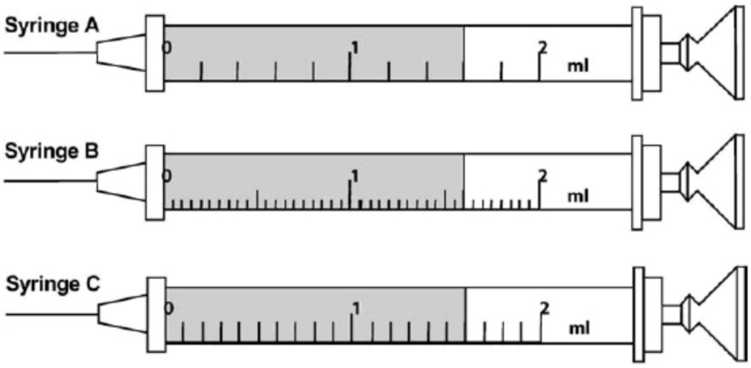

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- **Listen before intervening**
  When approaching a group, stand back and listen to the discussion before intervening. It is all too easy to interrupt a group with a predetermined agenda, diverting their attention from the ideas they are discussing. This is not only annoying and disruptive (for the group), it also prevents learners from concentrating.
- **Join in; don't judge**
  Try to join in as an equal member of the group rather than as an authority figure. When teachers adopt judgmental roles, learners tend to try to 'guess what's in the teacher's head' rather than try to think for themselves: "Do you want us to say what we think, or what we think you want us to say?".
- **Ask learners to describe, explain and interpret**
  The purpose of an intervention is to increase the depth of reflective thought. Challenge learners to describe what they are doing (quite easy), to interpret something ("Can you say what that means?") or to explain something ("Can you show us why you say that?").
- **Do not do the thinking for learners**
  Many learners are experts at making their teachers do the work. They know that, if they 'play dumb' long enough, then the teacher will eventually take over. Try not to fall for this. If a learner says that they cannot explain something, ask another learner in the group to explain, or ask the learner to choose some part of the problem that they **can** explain. Don't let them off the hook. When a learner asks the teacher a question, don't answer it (at least not straight away). Ask someone else in the group to answer.

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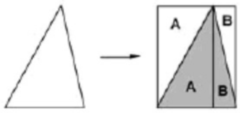
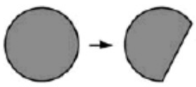
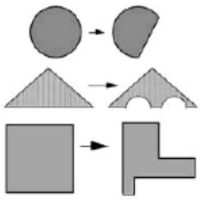
Statement cards	Hints cards
 Draw a triangle. There are three ways of drawing a rectangle so that it passes through all three vertices and shares an edge with the triangle. The areas of the three rectangles are equal.	 What fraction of each rectangle is the triangle? What happens when the triangle contains an obtuse angle?
 When you cut a piece off a shape you reduce its area and perimeter.	What happens to the area and perimeter with these cuts? 

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(v) Give feedback that is useful to learners

Evidence suggests that the only type of feedback that promotes learning is a meaningful comment (**not** a numerical score) on the quality of the work and constructive advice on how it should be improved. Indeed marks and grades usually detract learners from paying attention to qualitative advice. The research evidence [9] clearly shows that helpful feedback:

- focuses on the task, not on marks or grades;
- is detailed rather than general;
- explains **why** something is right or wrong;
- is related to objectives;
- makes clear what has been achieved and what has not;
- suggests what the learner may do next;
- describes strategies for improvement.

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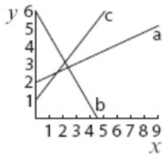
The Test



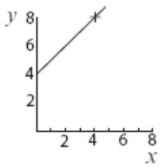
1. Find the equation for lines a, b and c.
2. In general, how do you find the equation from the graph?
3. Find the relationship in these sets of numbers and add another pair to the set:

A	1	2	4	6	10
B	1	7	19	31	55

4. Describe in not more than 50 words the definition of inverse.
5. Find the relationship:

C	−1	5	−5	0	2
D	3	9	−21	9	21

6. Given this rule, $4x - 2$, describe the numbers which go with (a) 3, (b) 6, (c) 10.
7. Find the equation of this line



8. Give an example of
   (a) a positive gradient
   (b) a negative gradient
   (c) a zero gradient
   Draw lines on the same set of axes.

The learners' test (left) and one learner's mark scheme (below) (including errors)

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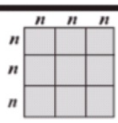
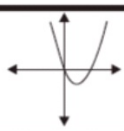
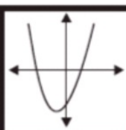
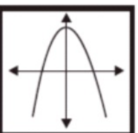
Interpreting algebraic notation

	
Square n then multiply your answer by 3	Multiply n by 3 then square your answer
$9n^2$	$(3n)^2$
$3n^2$	Square n then multiply your answer by 9

Interpreting calculus notation

		
$f(x) = 16 - x^2$	$f(0) = 0$	$f(0) > 0$
$f(x) = x^2 + 2x - 3$	$f'(0) = 0$	$f'(0) > 0$
$f(x) = x(x - 2)$	$f'(0) < 0$	$f(0) < 0$

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Collaborative Learning in Mathematics

Malcolm Swan
Shell Centre for Mathematics Education
School of Education
University of Nottingham
England

Introduction

Since 1979, I have conducted research and development with my colleagues at the University of Nottingham into more effective ways of teaching and learning mathematical concepts and strategies. This work is now coming to fruition through the dissemination of professional development resources: *Improving Learning in Mathematics* (Swan, 2005), *Thinking through Mathematics* (Swain & Swan, 2007; Swan & Wall, 2007) and *Bowland Maths* (Swan, 2008). These multimedia resources are now being distributed to schools and colleges across England.

These projects have similar aims. The first aim is to help students to adopt more active approaches towards learning. Our own research shows that many students view mathematics as a series of unrelated procedures and techniques that have to be committed to memory. Instead, we want them to engage in discussing and explaining ideas, challenging and teaching one another, creating and solving each other's questions and working collaboratively to share methods and results. The second aim is to develop more challenging, connected, collaborative orientations towards their teaching (Swan, 2005):

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Standards Unit

Improving learning in mathematics: challenges and strategies

Malcolm Swan

University of Nottingham